



# Synchronization of multiple chaotic FitzHugh–Nagumo neurons with gap junctions under external electrical stimulation

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## ABSTRACT

This paper discusses the synchronization of three coupled chaotic FitzHugh–Nagumo (FHN) neurons with different gap junctions under external electrical stimulation. A nonlinear control law that guarantees the asymptotic synchronization of coupled neurons (with reduced computations) is proposed. The developed control law incorporates the synchronization error between two slave neurons in addition to the conventionally considered synchronization errors between the master and the slave neurons, which make the proposed scheme computationally more efficient. Further, a novel  $L_2$  gain reduction criterion has been developed for multi-input multi-output systems with non-zero initial conditions, and is applied to robust synchronization of FHN neurons under  $L_2$  norm bounded disturbance and uncertainties. Furthermore, a robust adaptive nonlinear control law is developed, which is capable of handling variations in nonlinear part of synchronization error dynamics, without using any neural-network-based training-oriented adaptive scheme. The proposed control schemes ensure global synchronization with computational simplicity, easy way of design and implementation and avoiding extra measurements. The results obtained with the proposed control laws are verified through numerical simulations.

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## 1. Introduction

Dynamic behavior of neuron is widely studied to explore the chief role of neuronal spiking for effective neurotransmission and brain signal processing [1–5]. External electrical stimulation (EES), like deep brain stimulation, is a therapy for cognitive disorders such as Parkinson's disease, epilepsy and dystonia [6]. Synchronization of chaotic neurons under external stimulation plays a major role in the transmission of neural signals and enables efficient communication between the brain and the muscles [7,8]. Investigation of neuronal synchronization, for key understanding of the neural circuit functioning and the brain information processing, has become one of the widely studied problems in the field of neuroscience [9–12]. It has attracted many brain researchers over the past decade in order to understand the underlying mechanism of external stimulation and hence to improve the stimulation therapy based treatments for

cognitive diseases. FitzHugh–Nagumo (FHN) neural model, under sinusoidal electrical stimulation, is widely utilized for the synchronization study due to its potential ability of representing dynamical aspects of neurons [13–15].

Researchers have used various control strategies for synchronization of different models of neural networks without gap junctions [14–16]. Gap junctions are protein channels by which neurons communicate with each other [10,17–19]. Incorporation of the strength of gap junctions into the dynamics of coupled chaotic FHN neurons renders derivation of a control law for their synchronization difficult [20–22]. Some researchers have incorporated gap junctions for studying the behavior of two coupled chaotic neurons [23], and, in fact, various control strategies have been used for synchronization of two chaotic neurons with gap junctions. For instance in [20,21], synchronization has been achieved by canceling the gap junction terms present in FHN neuron dynamics, which cannot be possible due to variation in the strength of gap junctions. Moreover, the control laws reported recently for two neurons with gap junctions [20,21] are computationally complex. Two control inputs have been utilized for a single slave neuron, which is not applicable in reality, making the technique [20] still more conservative.

Dynamical behavior, network architecture and synchronization of multiple interacting chaotic FHN neurons with gap junctions have been although addressed in the literature [18,24], but

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still, the development of control strategies for synchronization of multiple coupled chaotic FHN neurons with different and uncertain gap junctions remained elusively rare up to this date. Synchronization of multiple interlinked chaotic neurons with different and uncertain gap junctions is challenging due to their inherently complex coupling. Nonetheless, development of a control law for synchronization of a network of FHN neurons is tantalizingly attractive owing to its potential utility for restoration of effective neuro-system communication under EES. Most recently, synchronization of multiple oscillators has been discussed in [25], but much detailed work remains to be done.

In this paper, we address the issue of synchronization of three coupled chaotic FHN neurons with gap junctions, by applying nonlinear, robust and robust adaptive control strategies. A novel  $L_2$  gain reduction criterion for systems with non-zero initial conditions has been developed and applied to synchronize FHN neurons under  $L_2$  norm bounded disturbance. As gap junctions represent the properties of the medium between any two connected neurons, which differ for each of the connection, we take different and uncertain gap junctions between any two interlinked FHN neurons and ensure robust synchronization of uncertain chaotic neurons. The proposed control schemes are computationally more efficient than the results reported in [20–22], and utilize a single control input for each of the slave neurons. Moreover, consideration of controller design without canceling the gap junction terms makes the proposed strategies less conservative than the conventional methods. Our schemes consider synchronization of three FHN neurons, which is helpful for synchronization of a network of FHN neurons with known neural connections.

In order to deal with model variations associated with non-linear part of neural dynamics, a computationally efficient robust adaptive control law has also been developed (see also [26–30]). This robust adaptive control strategy guarantees asymptotic synchronization of three interconnected FHN neurons and ensures the  $L_2$  gain reduction from disturbance to synchronization error. The proposed control schemes are utilizing linear matrix inequalities (LMI's) for computing large matrices in order to reduce parameter tuning efforts [31–34]. The main contribution of this paper is described below:

- (1) This paper proposes a novel  $L_2$  gain reduction criterion from input to output of a multi-input multi-output (MIMO) continuous-time system with non-zero initial condition and provides its application for synchronization of chaotic FHN neurons under disturbance. The alternative  $l_2$  gain reduction criterion for discrete-time systems is also reported.
- (2) This paper describes nonlinear, robust and robust adaptive control schemes for synchronization of three coupled chaotic FHN neurons with uncertain and different strengths of gap junctions. This work provides an understanding of issues for synchronization of multiple FHN neurons.
- (3) The proposed control strategies are computationally efficient, global, easy to design and implement, avoid extra measurements and reduce parameter tuning efforts, hence suitable for implementation.

Simulation results that validate the proposed methodology are also presented. This work can be a step towards the synchronization control of a network of coupled FHN neurons with uncertain and different strengths of gap junctions.

This paper is organized as follows. Section 2 provides the  $L_2$  and  $l_2$  gain reduction criteria for MIMO systems with non-zero initial conditions. Section 3 presents the FHN model of three neurons coupled with different and uncertain gap junctions. Section 4 demonstrates the computationally efficient nonlinear

and robust nonlinear control laws for synchronization of coupled neurons. Section 5 discusses the proposed robust adaptive nonlinear control law for neuronal synchronization along with the pertinent simulation results. Section 6 draws the conclusions.

*Notations:* We use standard notations. The  $L_2$  norm of a vector  $d(t)$  is represented by

$$\|d(t)\|_2 = \left( \int_0^\infty \|d(t)\|^2 dt \right)^{1/2},$$

where  $\|d(t)\|$  represents the Euclidean norm of  $d(t)$  and  $t$  represents time. The  $L_2$  gain from a vector  $d(t)$  to another vector  $z(t)$  is represented by  $\sup_{\|d\|_2 \neq 0} (\|z(t)\| / \|d(t)\|_2)$ . For discrete-time systems, the  $l_2$  norm of a vector  $d(n)$  is represented by

$$\|d(n)\|_{l_2} = \left( \sum_{t=0}^{t=\infty} \|d(n)\|^2 \right)^{1/2},$$

where  $n$  denotes the  $n$ th sample; and the  $l_2$  gain from a vector  $d(n)$  to another vector  $z(n)$  is represented by  $\sup_{\|d\|_{l_2} \neq 0} (\|z(n)\|_{l_2} / \|d(n)\|_{l_2})$ . Identity and null matrices of appropriate dimensions are represented by  $I$  and  $O$ , respectively.

## 2. $L_2$ and $l_2$ gain reduction criteria

Many physical systems are affected by disturbances and noises that can be bounded in  $L_2$  norm sense. The main objective, for stabilization, regulation, tracking and synchronization of physical systems is to minimize the effect of  $L_2$  norm bounded disturbances and noises at output. However, the traditional  $L_2$  gain reduction criteria [35–37], from input to output of a system, are based on the supposition of zero initial condition. The aim of this section is to provide a novel  $L_2$  gain reduction criterion from input to output of general MIMO continuous-time control systems having non-zero initial conditions. Additionally, due to significance of discrete-time control systems, the counterpart of this criterion in terms of  $l_2$  gain reduction has also been presented.

Consider a general class of continuous-time MIMO systems given by

$$\dot{s} = f(s, w, t, \tau), \quad z = g(s, w, t, \tau) \tag{1}$$

where  $s(t) \in R^m$ ,  $w(t) \in R^p$  and  $z(t) \in R^q$  represent the vectors for state, input and output, respectively;  $f(s, w, t, \tau) \in R^m$  and  $g(s, w, t, \tau) \in R^q$ . Here,  $t$  and  $\tau$  represent time and time-delay, respectively. Note that the delay  $\tau$  can also be zero for systems without time-delay. We take the following assumption:

**Assumption 1.** The  $L_2$  norm of input  $w$  is bounded by  $w_m$ , that is  $\|w\|_2 \leq w_m$ , where  $w_m$  is a positive scalar.

To address the  $L_2$  gain reduction from input to output of a MIMO system (1) with any initial condition, we introduce the following important lemma.

**Lemma 1.** Consider the system (1) satisfying Assumption 1. Suppose that there exists a Lyapunov function  $E(s, t) > 0$  (or  $E(s, t, \tau) > 0$ ) such that

$$\dot{E}(s, t) + z^T z - \gamma^2 w^T w < 0, \tag{2}$$

where  $\gamma > 0$  is a scalar. Then, the following are ensured.

- (i) The system (1) is asymptotically stable if  $w(t) = 0$ .
- (ii) The output  $z$  satisfies  $\|z(t)\|_2^2 < (\gamma^2 + (E(s, 0)/w_m^2)) \|w(t)\|_2^2$ , if  $w(t) \neq 0$ . In other words, the  $L_2$  gain from  $w(t)$  to  $z(t)$  is less than  $\sqrt{\gamma^2 + (E(s, 0)/w_m^2)}$ .

**Proof.** If  $\dot{E}(s,t) + z^T z - \gamma^2 w^T w < 0$ , then the following two cases hold:

- (a) If  $w(t) = 0$ , then  $\dot{E}(s,t) + z^T z < 0$ , which implies that  $\dot{E}(s,t) < 0$ . Hence the system (1) is asymptotically stable.
- (b) If  $w(t) \neq 0$ , then, integrating (2) from 0 to  $T_o \rightarrow \infty$ , we get

$$\lim_{T_o \rightarrow \infty} \left( E(s, T_o) - E(s, 0) + \int_0^{T_o} z^T z dt - \gamma^2 \int_0^{T_o} w^T w dt \right) < 0. \quad (3)$$

As  $E(s, T_o) > 0$ , (3) implies

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 - E(s, 0) < 0, \quad (4)$$

which can be written as

$$\left( \|z\|_2^2 / \|w\|_2^2 - \gamma^2 - (E(s, 0) / \|w\|_2^2) \right) < 0 \quad (5)$$

Using  $\|w\|_2 \leq w_m$ , we get

$$\left( \|z\|_2^2 / \|w\|_2^2 - \gamma^2 - (E(s, 0) / w_m^2) \right) < 0, \quad (6)$$

which implies that  $\|z(t)\|_2^2 < (\gamma^2 + (E(s, 0) / w_m^2)) \|w(t)\|_2^2$ . Hence, the  $L_2$  gain from input  $w(t)$  to output  $z(t)$  is less than  $\sqrt{\gamma^2 + (E(s, 0) / w_m^2)}$ , which completes the proof of Lemma 1. Note that (6) also implies

$$\left( \|z\|_2^2 / \|w\|_2^2 - \gamma^2 - (E(s, 0) / w_m^2) - 2\gamma \sqrt{E(s, 0) / w_m^2} \right) < 0, \quad (7)$$

which states that the  $L_2$  gain from  $w(t)$  to  $z(t)$  is also less than  $(\gamma + \sqrt{E(s, 0) / w_m^2})$ .  $\square$

**Remark 1.** In the past (for instance, see [35–38]), a large number of research papers used inequalities like  $\dot{E}(s,t) + z^T z - \gamma^2 w^T w < 0$  for stabilization, regulation, tracking and synchronization of linear, nonlinear, continuous-time, discrete-time and time-delay systems, assuming  $s(0) = 0$  (or  $s(t - \tau) = 0, \forall t \leq \tau$ , for time-delay systems). It ensures that the  $L_2$  gain from  $w(t)$  to  $z(t)$  is less than  $\gamma$ . However, Lemma 1 shows that the inequality  $\dot{E}(s,t) + z^T z - \gamma^2 w^T w < 0$ , providing the  $L_2$  gain from  $w(t)$  to  $z(t)$  less than  $\sqrt{\gamma^2 + (E(s, 0) / w_m^2)}$ , is also well-applicable for MIMO systems (1) with non-zero initial conditions. The results proposed by Lemma 1 are more general, because if  $E(s, 0) = 0$  (corresponding to  $s(0) = 0$ ), the upper bound on  $L_2$  gain from  $w(t)$  to  $z(t)$  becomes  $\gamma$ . It is interesting to note that the  $L_2$  gain also depends on the initial condition of a system because  $E(s, 0)$  depends on  $s(0)$ .

**Remark 2.** The minimization of the effect of input  $w(t)$  at output  $z(t)$  can be achieved by minimizing the  $L_2$  gain  $\sqrt{\gamma^2 + (E(s, 0) / w_m^2)}$  or  $(\gamma + \sqrt{E(s, 0) / w_m^2})$ , which requires the minimization of  $\gamma$  and  $E(s, 0)$ . Here, the minimization of  $E(s, 0)$  depends on the selection of a Lyapunov function. For most frequently used quadratic Lyapunov function  $E(s,t) = s^T(t) P s(t)$  with  $P = P^T > 0$ , we can introduce a new inequality  $s^T(0) P s(0) \leq \mu s^T(0) s(0)$  with scalar  $\mu > 0$  and minimize  $E(s, 0)$  by minimizing  $\mu$ . If the bound  $w_m$  and  $s(0)$  are known, then  $\sqrt{\gamma^2 + (\mu s^T(0) s(0) / w_m^2)}$  can be minimized to obtain the optimal value of  $\sqrt{\gamma^2 + (s^T(0) P s(0) / w_m^2)}$ , otherwise a weighted combination of  $\gamma$  (or  $\bar{\gamma} = \gamma^2$ ) and  $\mu$  can be used for optimization. We suggest to use more weight for  $\gamma$  (or  $\bar{\gamma}$ ) in optimization because  $\mu$  is associated with the Lyapunov function that has to satisfy other constraints like stability, convergence rate, performance and robustness. It is also worth mentioning that if initial condition  $s(0)$  is not exactly known, then we can introduce a bound on initial condition. For instance, the upper bound on the  $L_2$  gain becomes  $\sqrt{\gamma^2 + \eta / w_m^2}$  if we confine the initial condition within an ellipsoid  $s(0) \in s^T(0) P s(0) \leq \eta$  for a positive scalar  $\eta$ .

In discrete-time systems, the  $l_2$  gain reduction from input to output is used for minimizing the effects of unwanted noises and disturbances (for example, see [38]). Now, we provide the counterpart of Lemma 1 due to wide applicability of the  $l_2$  gain reduction in discrete-time control systems. Consider the MIMO discrete-time system given by

$$s(n+1) = f(s, w, n), \quad z(n) = g(s, w, n). \quad (8)$$

**Assumption 2.** The  $l_2$  norm of input  $w(n)$  is bounded by a positive scalar  $w_m$ , that is  $\|w(n)\|_{l_2} \leq w_m$ .

**Lemma 2.** Consider the system (8) satisfying Assumption 2. Suppose that there exists a Lyapunov function  $E(s, n) > 0$  such that

$$E(s(n+1), n+1) - E(s(n), n) + z^T z - \gamma^2 w^T w < 0, \quad (9)$$

where  $\gamma > 0$  is a scalar, then the following are ensured:

- (i) The system (8) is asymptotically stable, if  $w(n) = 0$ .
- (ii) The output  $z$  satisfies  $\|z(n)\|_{l_2}^2 < (\gamma^2 + (E(s, 0) / w_m^2)) \|w(n)\|_{l_2}^2$ , if  $w(n) \neq 0$ . That is, the  $l_2$  gain from  $w(n)$  to  $z(n)$  is less than  $\sqrt{\gamma^2 + (E(s, 0) / w_m^2)}$ .

**Proof.** If (9) holds, then the following two cases hold:

- (a) If  $w(n) = 0$ , then  $E(s(n+1), n+1) - E(s(n), n) < 0$ . It implies that the system (8) is asymptotically stable.
- (b) If  $w(n) \neq 0$ , then, summing (9) from 0 to  $\infty$ , we get

$$\lim_{N \rightarrow \infty} \left( E(s(N), N) - E(s(0), 0) + \sum_{n=0}^{N-1} z^T z - \gamma^2 \sum_{n=0}^{N-1} w^T w \right) < 0. \quad (10)$$

Using a similar procedure as for the proof of Lemma 1, we get

$$\left( \|z\|_{l_2}^2 / \|w\|_{l_2}^2 - \gamma^2 - (E(s, 0) / w_m^2) \right) < 0. \quad (11)$$

Hence the  $l_2$  gain from  $w(n)$  to  $z(n)$  is less than  $\sqrt{\gamma^2 + (E(s, 0) / w_m^2)}$ , which completes the proof of Lemma 2. For  $l_2$  gain optimization, the same method described by Remark 2 can be used.  $\square$

### 3. FHN model description

Consider the following model of three coupled chaotic FHN neurons [18,39] with different gap junctions:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(x_1 - 1)(1 - rx_1) - y_1 - \tilde{g}_{12}(x_1 - x_2) - \tilde{g}_{13}(x_1 - x_3) \\ &\quad + (a/\omega)\cos\omega t + d_1, \\ \frac{dy_1}{dt} &= bx_1, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{dx_2}{dt} &= x_2(x_2 - 1)(1 - rx_2) - y_2 - \tilde{g}_{12}(x_2 - x_1) - \tilde{g}_{23}(x_2 - x_3) \\ &\quad + (a/\omega)\cos\omega t + d_2, \\ \frac{dy_2}{dt} &= bx_2, \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{dx_3}{dt} &= x_3(x_3 - 1)(1 - rx_3) - y_3 - \tilde{g}_{13}(x_3 - x_1) - \tilde{g}_{23}(x_3 - x_2) \\ &\quad + (a/\omega)\cos\omega t + d_3, \\ \frac{dy_3}{dt} &= bx_3, \end{aligned} \quad (14)$$

where  $x$  and  $y$  represent the state variables of a neuron representing the activation potential and the recovery voltage, respectively;  $x_1$  and  $y_1$  are the states of the master FHN neuron,  $x_2$  and  $y_2$  are the states of the first slave FHN neuron, and  $x_3$  and  $y_3$  are the states of the second slave FHN neuron. Here  $\tilde{g}_{12}$ ,  $\tilde{g}_{13}$  and  $\tilde{g}_{23}$

represent the strengths of gap junctions between the master and the first slave neurons, between the master and the second slave neurons, and between the two slave neurons, respectively. Disturbances at the master, the first slave and the second slave neurons are represented by  $d_1$ ,  $d_2$  and  $d_3$ , respectively. The term  $(a/\omega)\cos\omega t$  represents the external stimulation current with time  $t$  and angular frequency  $\omega$ . In the present study, we use the angular frequency  $\omega$  and the amplitude  $a$  as dimensionless quantities as specified for FHN model [21,30,39].

Traditionally, researchers have considered the synchronization of identical neurons (for example, see [14–20,22–24]); however, these coupled neurons can in no way be identical. If slight differences in the strengths of gap junctions ( $\tilde{g}_{12}$ ,  $\tilde{g}_{13}$  and  $\tilde{g}_{23}$ ) are accounted, the coupled neurons (12)–(14) are not completely identical. Additionally, the parameters representing the strengths of gap junctions are uncertain due to the property variations of the medium between interlinked neurons. As synchronization of neurons is affected by the properties of gap junctions, we consider robust synchronization of FHN neurons under different and uncertain parameters ( $\tilde{g}_{12}$ ,  $\tilde{g}_{13}$  and  $\tilde{g}_{23}$ ). Due to these facts, the model (12)–(14), in contrast to [18], has different and uncertain parameters ( $\tilde{g}_{12}$ ,  $\tilde{g}_{13}$  and  $\tilde{g}_{23}$ ) for each of the linkages among the

three neurons. We take

$$\tilde{g}_{12} = g_{12} + \Delta g_{12}, \quad \tilde{g}_{13} = g_{13} + \Delta g_{13}, \quad \tilde{g}_{23} = g_{23} + \Delta g_{23}, \quad (15)$$

where  $g_{12}$ ,  $g_{13}$  and  $g_{23}$  represent the nominal values and  $\Delta g_{12}$ ,  $\Delta g_{13}$  and  $\Delta g_{23}$  represent the uncertainties in strengths of gap junctions. We fix the parameters of the model as

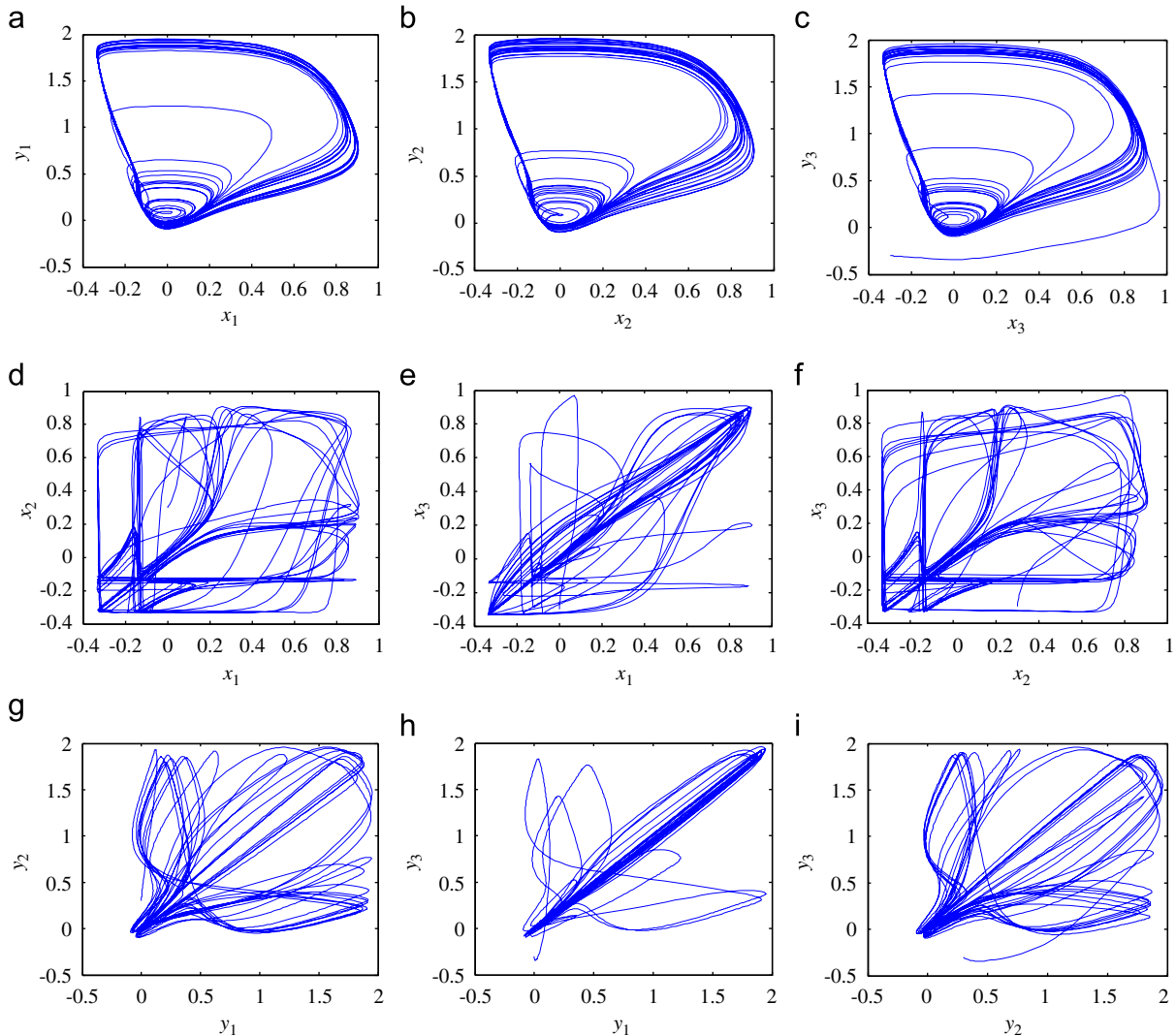
$$r = 10, \quad g_{12} = 0.011, \quad g_{13} = 0.012, \quad g_{23} = 0.013, \\ \omega = 0.254\pi, \quad b = 1 \quad \text{and} \quad a = 0.1, \quad (16)$$

and the initial condition as

$$x_1(0) = 0, \quad y_1(0) = 0, \quad x_2(0) = 0.3, \quad y_2(0) = 0.3, \\ x_3(0) = -0.3 \quad \text{and} \quad y_3(0) = -0.3. \quad (17)$$

Fig. 1 shows state-space plots of the three coupled chaotic FHN neurons under zero uncertainties and disturbances (see also [18,39]). Note that all three neurons are non-synchronous since the plots in Fig. 1(d)–(i) do not make straight lines of slope 1 passing through the origin.

To synchronize three FHN neurons, we use two control inputs  $u_1$  and  $u_2$  for the first and second slave neurons, respectively. The coupled nonlinear model (12)–(14) with two control inputs is,



**Fig. 1.** Chaotic behavior of coupled FHN neurons with gap junctions in EES. All three neurons are non-synchronous: (a) state space of  $x_1$  and  $y_1$ , (b) state space of  $x_2$  and  $y_2$ , (c) state space of  $x_3$  and  $y_3$ , (d) state space of  $x_1$  and  $x_2$ , (e) state space of  $x_1$  and  $x_3$ , (f) state space of  $x_2$  and  $x_3$ , (g) state space of  $y_1$  and  $y_2$ , (h) state space of  $y_1$  and  $y_3$  and (i) state space of  $y_2$  and  $y_3$ .

then, given by

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(x_1-1)(1-rx_1)-y_1-\tilde{g}_{12}(x_1-x_2)-\tilde{g}_{13}(x_1-x_3) \\ &\quad + (a/\omega)\cos\omega t + d_1, \\ \frac{dy_1}{dt} &= bx_1, \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{dx_2}{dt} &= x_2(x_2-1)(1-rx_2)-y_2-\tilde{g}_{12}(x_2-x_1)-\tilde{g}_{23}(x_2-x_3) \\ &\quad + (a/\omega)\cos\omega t + u_1 + d_2, \\ \frac{dy_2}{dt} &= bx_2, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{dx_3}{dt} &= x_3(x_3-1)(1-rx_3)-y_3-\tilde{g}_{13}(x_3-x_1)-\tilde{g}_{23}(x_3-x_2) \\ &\quad + (a/\omega)\cos\omega t + u_2 + d_3, \\ \frac{dy_3}{dt} &= bx_3. \end{aligned} \tag{20}$$

The goal of the present study is to design proper control signals  $u_1$  and  $u_2$  for synchronization of both the slave neurons with the master neuron.

#### 4. Robust nonlinear control

This section describes a nonlinear state feedback control law for synchronization of three FHN neurons (18)–(20), by selecting control signals  $u_1$  and  $u_2$  to achieve  $x_2, x_3 \rightarrow x_1$  and  $y_2, y_3 \rightarrow y_1$ . A robust nonlinear synchronization control law has also been developed against model uncertainties and disturbances. The proposed control law for synchronization of three coupled chaotic FHN neurons is given by

$$u_1 = C_o(x_1-x_2)-((1+r)x_2^2-rx_2^3)+((1+r)x_1^2-rx_1^3), \tag{21}$$

$$u_2 = C_o(x_1-x_3)-((1+r)x_3^2-rx_3^3)+((1+r)x_1^2-rx_1^3), \tag{22}$$

where  $C_o$  is a constant to be determined. Similar to [40], two linear feedback terms  $C_o(x_1-x_2)$  and  $C_o(x_1-x_3)$  are used to ensure the convergence of the states of two slave neurons to the state of the master neuron; while nonlinear feedback terms  $((1+r)x_1^2-rx_1^3)$ ,  $((1+r)x_2^2-rx_2^3)$  and  $((1+r)x_3^2-rx_3^3)$  are used to cancel the nonlinear terms in synchronization error dynamics, which will be discussed later. Now we provide a sufficient condition for synchronization of the three FHN neurons (18)–(20) using control laws (21) and (22) in the absence of uncertainties and disturbances.

**Theorem 1.** Consider the coupled FHN neurons under  $d_1=d_2=d_3=0$  and  $\Delta g_{12}=\Delta g_{13}=\Delta g_{23}=0$ . The nonlinear control laws ensure the asymptotic synchronization of three neurons if the following matrix inequalities are satisfied:

$$P > 0, \quad A^T P + P A < 0, \tag{23}$$

where

$$A = \begin{bmatrix} -(1+C_o+2g_{12}) & -g_{13} & g_{23} & -1 & 0 & 0 \\ -g_{12} & -(1+C_o+2g_{13}) & -g_{23} & 0 & -1 & 0 \\ g_{12} & -g_{13} & -(1+C_o+2g_{23}) & 0 & 0 & -1 \\ b & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

and  $P \in R^{6 \times 6}$  is a symmetric matrix.

**Proof.** Incorporating (21) and (22) into (19) and (20), the overall closed-loop system becomes

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(x_1-1)(1-rx_1)-y_1-\tilde{g}_{12}(x_1-x_2)-\tilde{g}_{13}(x_1-x_3) \\ &\quad + (a/\omega)\cos\omega t + d_1, \\ \frac{dy_1}{dt} &= bx_1, \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{dx_2}{dt} &= \left\{ -rx_1^3 + (1+r)x_1^2 - x_2 - y_2 - \tilde{g}_{12}(x_2-x_1) \right. \\ &\quad \left. - \tilde{g}_{23}(x_2-x_3) + (a/\omega)\cos\omega t + C_o(x_1-x_2) + d_2 \right\}, \\ \frac{dy_2}{dt} &= bx_2, \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{dx_3}{dt} &= \left\{ -rx_1^3 + (1+r)x_1^2 - x_3 - y_3 - \tilde{g}_{13}(x_3-x_1) \right. \\ &\quad \left. - \tilde{g}_{23}(x_3-x_2) + (a/\omega)\cos\omega t + C_o(x_1-x_3) + d_3 \right\}, \\ \frac{dy_3}{dt} &= bx_3. \end{aligned} \tag{27}$$

The following synchronization errors are taken into account.

$$e_1 = x_1 - x_2, \quad e_2 = x_1 - x_3, \quad e_3 = x_2 - x_3, \tag{28}$$

$$e_4 = y_1 - y_2, \quad e_5 = y_1 - y_3 \quad \text{and} \quad e_6 = y_2 - y_3. \tag{29}$$

The purpose of control laws (21) and (22) is to ensure the convergence of synchronization errors (28) and (29) to zero for a proper selection of  $C_o$ . The derivatives of the error terms in (28) and (29) along (25)–(27) yield the synchronization error dynamics:

$$\frac{de_1}{dt} = -(1+C_o+2\tilde{g}_{12})e_1 - \tilde{g}_{13}e_2 + \tilde{g}_{23}e_3 - e_4 + d_1 - d_2, \tag{30}$$

$$\frac{de_2}{dt} = -\tilde{g}_{12}e_1 - (1+C_o+2\tilde{g}_{13})e_2 - \tilde{g}_{23}e_3 - e_5 + d_1 - d_3, \tag{31}$$

$$\frac{de_3}{dt} = \tilde{g}_{12}e_1 - \tilde{g}_{13}e_2 - (1+C_o+2\tilde{g}_{23})e_3 - e_6 + d_2 - d_3, \tag{32}$$

$$\frac{de_4}{dt} = be_1, \tag{33}$$

$$\frac{de_5}{dt} = be_2, \tag{34}$$

$$\frac{de_6}{dt} = be_3, \tag{35}$$

which can be rewritten in the state space form as

$$\frac{de}{dt} = Ae + \Delta Ae + B\tilde{d}, \tag{36}$$

where

$$e = [e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6]^T, \tag{37}$$

$$\Delta A = \begin{bmatrix} \Delta A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{38}$$

$$\Delta A_{11} = \begin{bmatrix} -2\Delta g_{12} & -\Delta g_{13} & \Delta g_{23} \\ -\Delta g_{12} & -2\Delta g_{13} & -\Delta g_{23} \\ \Delta g_{12} & -\Delta g_{13} & -2\Delta g_{23} \end{bmatrix}, \tag{39}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \tag{40}$$

$$\tilde{d} = [d_1 - d_2 \quad d_1 - d_3 \quad d_2 - d_3]^T, \tag{41}$$

and  $A$  has been already defined in (24). From (36), it is clear that the synchronization error dynamics is linear, due to the cancelation of



the nonlinear terms in the model with those in the control law. Now consider the following quadratic Lyapunov function candidate:

$$E(e,t) = e^T \gamma Pe, \tag{42}$$

where  $P > 0$ , and we take  $\gamma = 1$  for the present case. The derivative of (42) along (36) becomes

$$\dot{E}(e,t) = e^T \gamma (A^T P + PA)e + e^T \gamma (\Delta A^T P + P \Delta A)e + \gamma e^T P B \tilde{d} + \gamma \tilde{d}^T B^T P e. \tag{43}$$

Using  $\tilde{d} = 0$ ,  $\Delta A = 0$  and  $\gamma = 1$ , and  $A^T P + PA < 0$ ,  $\dot{E}(e,t) < 0$  is concluded, which completes the proof of Theorem 1.  $\square$

**Remark 3.** The proposed control laws (21) and (22) are much simpler than that in [20–22], as it does not include the gap junctions and terms like  $y_1 - y_2$  and  $y_1 - y_3$ , among others. Moreover, only a single control input is used for each slave neuron to achieve a computationally efficient control law. Furthermore, this control law does not require extra measurements of recovery voltages. Such a control law is advantageous for real-time synchronization of multiple chaotic coupled FHN neurons.

**Remark 4.** In the literature [14–16,19–22,25,30,31], synchronization errors only between the master and slave neurons were considered. In the present scenario, however, the synchronization errors between two slave neurons ( $e_3, e_6$ ) are considered in addition to those between the master and the slave neurons ( $e_1, e_2, e_4$  and  $e_5$ ). Note that the convergence of  $e_1, e_2, e_4$  and  $e_5$  to zero ensures the convergence of  $e_3$  and  $e_6$ . This implies that  $e_3$  and  $e_6$  should not be considered in (28) and (29) as synchronization errors. However, if in the control law derivation, state differences in slave neurons ( $e_3, e_6$ ) are not used, one has to incorporate redundant terms in control signals (21) and (22) for cancellation of terms  $g_{23}e_3, -g_{23}e_3$  and  $-e_6$  present in (30)–(32). Hence, for the synchronization of multiple neurons, it is vital to consider the synchronization errors between the states of any two slave neurons for obtaining a simple controller with less computational complexity.

The main problem for synchronization of chaotic systems under  $L_2$  disturbance is that the initial condition cannot be zero (that is,  $e(t) \neq 0$  at  $t=0$ ), while the traditional controller design techniques are based on the assumption  $e(0)=0$ . This problem can be resolved by applying Lemma 1. To address the robust synchronization of FHN neurons under disturbance and uncertainty, we take the following assumptions:

**Assumption 3.** The  $L_2$  norm of disturbance  $\tilde{d}$  is bounded as  $\|\tilde{d}\|_2 \leq \tilde{d}_m$ , where  $\tilde{d}_m$  is a positive scalar.

**Assumption 4.** The uncertain terms  $\Delta g_{12}, \Delta g_{13}$  and  $\Delta g_{23}$  are bounded as  $|\Delta g_{12}| < g_m, |\Delta g_{13}| < g_m$  and  $|\Delta g_{23}| < g_m$ , where  $g_m$  is a positive scalar.

Now, by virtue of Lemma 1, we provide a sufficient condition for robust synchronization of three FHN neurons (18)–(20) under bounded disturbance and parametric uncertainties.

**Theorem 2.** Consider the three coupled FHN neurons satisfying Assumption 3–4. Suppose the optimization problem:

$$\begin{aligned} & \min c_1 \gamma + c_2 \mu \\ & \text{such that} \\ & P = P^T > 0, \quad P < \mu I, \quad \gamma > 0, \quad \varepsilon > 0, \end{aligned} \tag{44}$$

$$\psi = \begin{bmatrix} A^T P + PA + \varepsilon M & PB & I & P \\ * & -\gamma & 0 & 0 \\ * & * & -\gamma & 0 \\ * & * & 0 & -\varepsilon \end{bmatrix} < 0, \tag{45}$$

with

$$M = \begin{bmatrix} M_{11} & O \\ O & O \end{bmatrix}, \tag{46}$$

$$M_{11} = g_m^2 \begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 3 \\ -3 & 3 & 6 \end{bmatrix}, \tag{47}$$

where  $\mu, \gamma$  and  $\varepsilon$  are scalars,  $P$  is a symmetric matrix and scalars  $c_1$  and  $c_2$  are the optimization weights. The nonlinear control laws ensure the following:

- (i) Asymptotic synchronization of the FHN neurons, if  $\tilde{d} = 0$ .
- (ii) The  $L_2$  gain from  $\tilde{d}$  to  $e$  is less than  $\sqrt{\gamma^2 + (\gamma e(0)^T P e(0) / \tilde{d}_m^2)}$ .

**Proof.** Consider the following inequalities:

$$E(e,t) = e^T \gamma Pe < \gamma \mu e^T e, \tag{48}$$

$$\dot{E}(e,t) + e^T e - \gamma^2 \tilde{d}^T \tilde{d} < 0. \tag{49}$$

Here (48) ensures  $P < \mu I$ . By application of Lemma 1 for Lyapunov function (42), the inequality (49) ensures asymptotic synchronization of FHN neurons if  $\tilde{d} = 0$  and the  $L_2$  gain from  $\tilde{d}$  to  $e$  is less than  $\sqrt{\gamma^2 + (\gamma e(0)^T P e(0) / \tilde{d}_m^2)}$ . Using (43) into (49), we get

$$e^T (A^T P + PA)e + e^T (\Delta A^T P + P \Delta A)e + e^T P B \tilde{d} + \tilde{d}^T B^T P e + \gamma^{-1} e^T e - \gamma \tilde{d}^T \tilde{d} < 0. \tag{50}$$

Using (38), (39), (46), (47) and Assumption 4, it is trivial to obtain

$$\Delta A^T \Delta A = \begin{bmatrix} \Delta A_{11}^T \Delta A_{11} & O \\ O & O \end{bmatrix} \leq M, \tag{51}$$

due to the specific structure of  $\Delta A$  in (38) and (39). For any scalar  $\varepsilon > 0$  and using (51), we obtain

$$\Delta A^T P + P \Delta A \leq \varepsilon \Delta A^T \Delta A + \varepsilon^{-1} P^2 \leq \varepsilon M + \varepsilon^{-1} P^2. \tag{52}$$

Incorporating (52) into (50), we get

$$e^T (A^T P + PA + \varepsilon M + \varepsilon^{-1} P^2 + \gamma^{-1} I)e + e^T P B \tilde{d} + \tilde{d}^T B^T P e - \gamma \tilde{d}^T \tilde{d} < 0, \tag{53}$$

which is equivalent to

$$\begin{bmatrix} A^T P + PA + \varepsilon M + \varepsilon^{-1} P^2 + \gamma^{-1} I & PB \\ * & -\gamma \end{bmatrix} < 0. \tag{54}$$

By applying the Schur complement [41,42], we obtain the matrix inequalities (44) and (45), which complete the proof of Theorem 2  $\square$ .

### 5. Robust adaptive control

In the previous section, a nonlinear robust control law, based on cancelation of nonlinear terms in synchronization error dynamics, has been developed. However, due to uncertainty in the nonlinear part of FHN neurons, exact cancelation cannot be possible. This section addresses a robust adaptive control law to ensure synchronization of neurons under uncertainty in the nonlinear part of neural dynamics, in addition to  $L_2$  bounded disturbance and uncertainties in gap junctions. The classical way of dealing with the nonlinear part uncertainty is to develop a control law by considering whole nonlinear function as an uncertainty. For instance, the adaptation law [22], approximating whole nonlinear function using a neural-network-based approach, is however computationally complex due

to the use of a number of neural nodes. Moreover, such synchronization techniques, requiring additional software for training of the neural networks, are capable to provide the local synchronization due to the use of local approximation of nonlinear functions. The other less conservative way is to adapt uncertain parameter  $r$  present in the nonlinear part of FHN neurons (see [43–45]). In this section, we present a computationally simpler continuous-time robust adaptive control law ensuring global synchronization of three FHN neurons. The proposed control law is given by

$$u_1 = C_0(x_1 - x_2) - ((1 + \hat{r})x_2^2 - \hat{r}x_2^3) + ((1 + \hat{r})x_1^2 - \hat{r}x_1^3), \tag{55}$$

$$u_2 = C_0(x_1 - x_3) - ((1 + \hat{r})x_3^2 - \hat{r}x_3^3) + ((1 + \hat{r})x_1^2 - \hat{r}x_1^3), \tag{56}$$

where  $\hat{r}$  is a time varying adaptive parameter of the proposed control law.

**Assumption 5.** The parameter  $r$  is bounded by a positive value  $r_m$ . That is

$$|r| < r_m. \tag{57}$$

The adaptation law for  $\hat{r}$  is given by

$$\dot{\hat{r}} = -0.5(e^T PF(x) + F^T(x)Pe)S, \tag{58}$$

where  $S$  is a positive scalar, and

$$F(x) = [f_1 \quad f_2 \quad f_3 \quad 0 \quad 0 \quad 0]^T, \tag{59}$$

with

$$f_1 = (x_2^2 - x_2^3 - x_1^2 + x_1^3), \tag{60}$$

$$f_2 = (x_3^2 - x_3^3 - x_1^2 + x_1^3), \tag{61}$$

$$f_3 = (x_3^2 - x_3^3 - x_2^2 + x_2^3). \tag{62}$$

**Theorem 3.** Consider the three coupled FHN neurons satisfying Assumptions 3–5. Suppose the following optimization problem:

$$\begin{aligned} \min \quad & c_1\gamma + c_2\mu \\ \text{such that} \quad & \\ P > 0, \quad & P < \mu I, \end{aligned} \tag{63}$$

$$\gamma > 0, \quad \varepsilon > 0, \quad \psi < 0, \tag{64}$$

where  $\mu$ ,  $\gamma$  and  $\varepsilon$  are scalars,  $P$  is a symmetric matrix and scalars  $c_1$  and  $c_2$  are the optimization weights. The nonlinear control law along with the adaptation law (58) ensures the following:

- (i) Asymptotic synchronization of the FHN neurons, if  $\tilde{d} = 0$ .
- (ii) The  $L_2$  gain from  $\tilde{d}$  to  $e$  is less than  $\sqrt{\gamma^2 + (\gamma e(0)^T Pe(0) / \tilde{d}_m^2)}$ .

**Proof.** Incorporating the control law (55) and (56) into (19) and (20), the closed-loop system becomes

$$\begin{aligned} \frac{dx_1}{dt} &= -rx_1^3 + (1+r)x_1^2 - x_1 - y_1 - \tilde{g}_{12}(x_1 - x_2) - \tilde{g}_{13}(x_1 - x_3) \\ &\quad + (a/\omega)\cos\omega t + d_1, \\ \frac{dy_1}{dt} &= bx_1, \end{aligned} \tag{65}$$

$$\begin{aligned} \frac{dx_2}{dt} &= -\hat{r}x_1^3 + (1+\hat{r})x_1^2 - (r-\hat{r})x_2^3 + (r-\hat{r})x_2^2 - x_2 - y_2 - \tilde{g}_{12}(x_2 - x_1) \\ &\quad - \tilde{g}_{23}(x_2 - x_3) + (a/\omega)\cos\omega t + C_0(x_1 - x_2) + d_2, \\ \frac{dy_2}{dt} &= bx_2, \end{aligned} \tag{66}$$

$$\begin{aligned} \frac{dx_3}{dt} &= -\hat{r}x_1^3 + (1+\hat{r})x_1^2 - (r-\hat{r})x_3^3 + (r-\hat{r})x_3^2 - x_3 - y_3 - \tilde{g}_{23}(x_3 - x_2) \\ &\quad - \tilde{g}_{23}(x_3 - x_2) + (a/\omega)\cos\omega t + C_0(x_1 - x_3) + d_3, \\ \frac{dy_3}{dt} &= bx_3. \end{aligned} \tag{67}$$

Using the same procedure as in the previous section, synchronization error dynamics for this case becomes

$$\frac{de}{dt} = Ae + \Delta Ae + F(x)(\hat{r} - r) + B\tilde{d}. \tag{68}$$

Consider the following Lyapunov function candidate [32,33]:

$$E = e^T \gamma Pe + (\hat{r} - r)^2 \gamma S^{-1}, \tag{69}$$

with  $\gamma > 0$ ,  $P > 0$  and  $S > 0$ . Taking the derivative of (69) along (68), we obtain

$$\begin{aligned} \dot{E}(e,t) &= \gamma(e^T(A^T P + PA)e + e^T(\Delta A^T P + P\Delta A)e + e^T PB\tilde{d} + \tilde{d}^T B^T Pe \\ &\quad + e^T PF(x)(\hat{r} - r) + F^T(x)Pe(\hat{r} - r) + 2(\hat{r} - r)\dot{\hat{r}}S^{-1}). \end{aligned} \tag{70}$$

Choosing  $e^T PF(x) + F^T(x)Pe + 2\dot{\hat{r}}S^{-1} = 0$  from the adaptation law (58) and (70) reduces to (43). Further using the same steps as in the proof of Theorem 2, we obtain the matrix inequalities (63) and (64), which complete the proof of Theorem 3.  $\square$

**Remark 5.** The proposed adaptation law (58) does not use any kind of neural networks. Hence, there is no training required, in contrast to [22]. Moreover, the proposed control law (55) and (56) and adaptation law (58) are computationally simpler than [22]. This makes the proposed control strategy suitable and indeed advantageous for practical implementation.

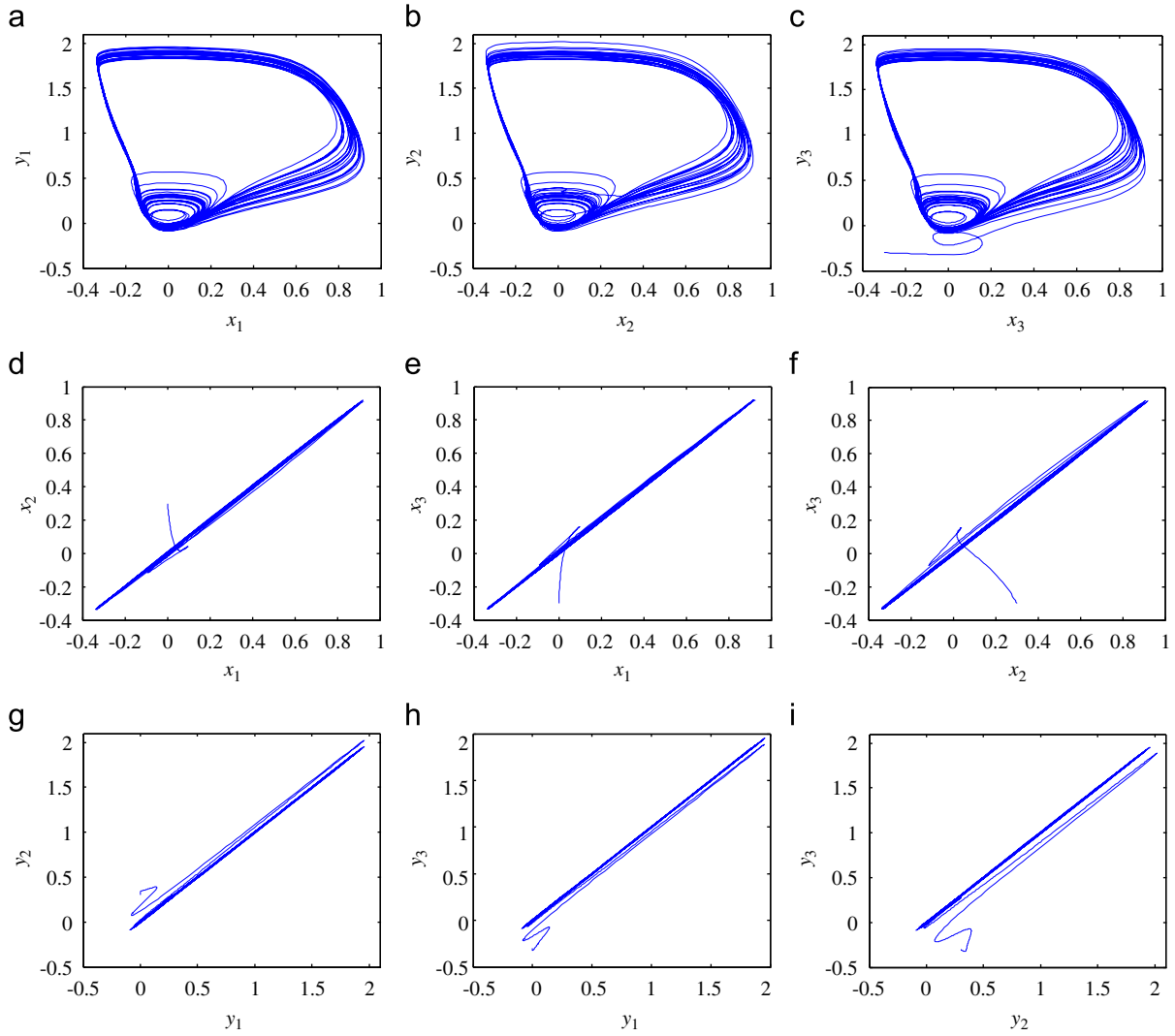
**Remark 6.** The approaches developed in the present study are global and based on the solution of LMI's for a proper selection of controller parameter  $C_0$ . In contrast to [15,22], the selection of large matrices can be made easily using LMI-based tools. It becomes necessary to use such LMI tools for selection of large matrices like  $P$  when dealing with synchronization of multiple coupled FHN neurons.

**Remark 7.** In [22], the robust adaptive technique developed for synchronization of two neurons separated by gap junctions is based on the convergence of synchronization errors within a small region rather than to zero, even for zero disturbance. The adaptive control proposed in the present work is less conservative because it ensures convergence of synchronization errors to zero for  $\tilde{d} = 0$ . Moreover, the adaptive-control-based synchronization of FHN neurons proposed in [14,20] does not provide any indication of stability. Furthermore, the selection of a large number of control parameters, with the aforementioned methodologies, is difficult.

To confirm the validity of the proposed control methodology, we select  $C_0 = 5$ . By solving the matrix inequalities of Theorem 2 or 3 for  $g_m = 0.2$ ,  $c_1 = 1$  and  $c_2 = 0.1$ , we obtain

$$P = \begin{bmatrix} 4.6333 & 0.1665 & -0.1639 & 0.9945 & -0.0246 & 0.0246 \\ 0.1665 & 4.6298 & 0.1614 & -0.0248 & 0.9946 & -0.0247 \\ -0.1639 & 0.1614 & 4.6262 & 0.0250 & -0.0250 & 0.9948 \\ 0.9945 & -0.0248 & 0.0250 & 6.2878 & 0.7104 & -0.7109 \\ -0.0246 & 0.9946 & -0.0250 & 0.7104 & 6.2890 & 0.7115 \\ 0.0246 & -0.0247 & 0.9948 & -0.7109 & 0.7115 & 6.2902 \end{bmatrix}, \tag{71}$$

$$\mu = 7.41, \quad \gamma = 1.34, \quad \varepsilon = 103.2. \tag{72}$$



**Fig. 2.** Synchronization of three coupled chaotic FHN neurons with uncertain and different gap junctions under disturbance using the proposed adaptive nonlinear control: (a) state space of  $x_1$  and  $y_1$ , (b) state space of  $x_2$  and  $y_2$ , (c) state space of  $x_3$  and  $y_3$ , (d) state space of  $x_1$  and  $x_2$ , (e) state space of  $x_1$  and  $x_3$ , (f) state space of  $x_2$  and  $x_3$ , (g) state space of  $y_1$  and  $y_2$ , (h) state space of  $y_1$  and  $y_3$  and (i) state space of  $y_2$  and  $y_3$ .

Fig. 2 shows the results, so obtained, by utilizing the proposed adaptive control. The initial condition of  $\hat{r}$  is taken to be zero, and  $\Delta g_{12}=0.1$ ,  $\Delta g_{13}=0.14$  and  $\Delta g_{23}=0.18$ . The disturbances are taken as

$$\begin{aligned} d_1 &= 0.02 \sin(t), \\ d_2 &= 0.02 \sin(1.1t), \\ d_3 &= 0.02 \sin(1.2t). \end{aligned} \tag{73}$$

It is evident that all three of the coupled chaotic FHN neurons are synchronized for identical behaviors because the plots in Fig. 2(d)–(i) make straight lines having slope 1 passing through the origin, which shows the robustness in the presence of disturbances and parametric uncertainties.

**6. Conclusions**

This paper discussed the synchronization of three coupled chaotic FHN neurons, in which one of them was considered as the master and the other two as slave neurons. The strengths of gap junctions between the neurons of each connection were assumed different and uncertain. A new  $L_2$  gain reduction criterion from input to output of a MIMO system with non-zero initial condition was

established and was effectively used for synchronization of FHN neurons under  $L_2$  norm bounded disturbance. Robust nonlinear/adaptive control laws for synchronization of FHN neurons were developed. The developed control laws are simple in implementation, avoiding additional measurements for recovery voltages and reducing parameter tuning efforts. The simulation results demonstrated the success and effectiveness of the overall scheme. The present work can be extended to the synchronization of a network of coupled FHN neurons with uncertain and different gap junctions.

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